THICCC Homework

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Week 5

Problem 0. Review the notes and convince yourself that we didn't make everything up.

The problems after this point consist of short exercises, joke problems, and challenges. Some of these problems warrant use of facts and theorems outside of basic[1] geometry that we didn't cover in the notes, and these problems are marked with a star.

Additionally, we've split them into sections. The problems in the first section don't involve any conics, and the ones in the second do.

1 Canonical Geometry

Problem 1. Let ABC be a triangle such that $AC = BC$, and points P and Q inside the triangle satisfy $\angle PAC = \angle ABQ$ and $\angle PBC = \angle BAQ$. Prove that C, P, and Q are collinear.

Problem 2. In ABC, let D and E be points on BC such that $\angle BAD = \angle EAC$. Prove that

$$\frac{BD}{DC} \cdot \frac{BE}{EC} = \left(\frac{AB}{AC}\right)^2$$

Problem 3 (Foreshadowing)**.** Let $ABCD$ be a quadrilateral such that no point is the orthocenter of the other three. Show that the nine-point circles of $\triangle ABC$, $\triangle ABD$, $\triangle ACD$, and $\triangle BCD$ intersect at a point.

Problem 4 (Simson Lines)**.** Let P be a point on the circumcircle of $\triangle ABC$. Show that,

(i) The feet of the altitudes from P to the sides of $\triangle ABC$ are collinear (i.e. P's pedal circle is degenerate). This is called the *Simson line* of P with respect to $\triangle ABC$.

(ii) If P' is the P-antipode (i.e. the point on the circumcircle on $\triangle ABC$ diametrically opposite to P), then the lines connecting the vertices to the isogonal conjugate of P' and the Simson line of P are all parallel.

Problem 5. Suppose Q is the center of the circumcircle of the reflection of P over the sides of AB, BC, and AC. Show that this is equivalent to Q and P being isogonal conjugates.

Problem 6. Let $\triangle ABC$ be a triangle, and let P and Q be isogonal conjugates with respect to $\triangle ABC$. Let O_P be the circumcenter of $\triangle PBC$, and let O_Q be the circumcenter of $\triangle QBC$, and let O be the circumcenter of $\triangle ABC$. Show that $OO_P \cdot OO_Q = OA^2$.

[1] In this context, we mean AMC12 and some AIME level geometry techniques (such as angle chasing, trig, power of a point, etc.)

2 Conical Geometry

Problem 7 (c6h2985157)**.** Let $\triangle ABC$ be a triangle, and let $\mathcal{C}$ be the locus of points P such that $\angle ABP = \angle ACP$. Prove that,

(i) $\mathcal{C}$ passes through A,

(ii) The tangent to $\mathcal{C}$ at A, the tangent to the circumcircle of $\triangle ABC$ at B, and the tangent to the circumcircle of $\triangle ABC$ at C concur.

Problem 8 (Optical Property)***.** Let $\mathcal{C}$ be a conic, and let ℓ be a line tangent to $\mathcal{C}$ at point P. Show that,

(i) if $\mathcal{C}$ is a circle with center O, then $OP \perp \ell$,

(ii) if $\mathcal{C}$ is an ellipse with foci F_1 and F_2, then ℓ is the external angle bisector of $\angle F_1PF_2$,

(iii) if $\mathcal{C}$ is an hyperbola with foci F_1 and F_2, then ℓ is the internal angle bisector of $\angle F_1PF_2$,

(iv) if $\mathcal{C}$ is a parabola with focus F and Q is the foot of the perpendicular from P to the directrix of $\mathcal{C}$, then ℓ is the internal angle bisector of $\angle FPQ$.

Problem 9 (Folklore)***.** Suppose that an ellipse is inscribed in $\triangle ABC$, i.e. it is tangent to all three sides. Show that its foci are isogonal conjugates with respect to $\triangle ABC$.

Problem 10 (RTPS 2022)**.** In regular heptagon $ABCDEFG$ with center O, prove that the angle between the asymptotes of the circumhyperbola of $GABOE$ is $60°$.