# THICCC Homework

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Week 5

**Problem 0.** Review the notes and convince yourself that we didn't make everything up.

Again, problems requiring additional knowledge are marked with a star.

## 1 Problems

**Problem 1.** Prove Fontené's three theorems (in order of difficulty):

- Fontené's second theorem: If a point $P$ moves on a fixed line through the circumcenter $O$, then its pedal circle passes through a fixed point on the nine-point circle.
- Fontené's third theorem: If $P$ and $Q$ are isogonal conjugates, then their pedal circle is tangent to the nine-point circle if and only if $P$, $Q$, and $O$ are collinear.
- Fontené's first theorem: Let $\triangle ABC$ be a triangle such that $L$, $M$, and $N$ are the midpoints of $BC$, $CA$, and $AB$, respectively. Furthermore, let $P$ be a point such that $D$, $E$, and $F$ are the feet of the altitudes from $BC$, $CA$, and $AB$. Finally, let $X$ be the intersection of lines $MN$ and $EF$, let $Y$ be the intersection of lines $NL$ and $FD$, and let $Z$ be the intersection of lines $LM$ and $DE$. Then, the lines $DX$, $EY$, and $FZ$, the nine-point circle of $\triangle ABC$, and the pedal circle of $P$ intersect at a point.

**Problem 2** (Anti-foreshadowing)**.** Prove the following problem from yesterday's homework:

Let $ABCD$ be a quadrilateral such that no point is the orthocenter of the other three. Show that the nine-point circles of $\triangle ABC$, $\triangle ABD$, $\triangle ACD$, and $\triangle BCD$ intersect at a point.

**Problem 3** (c6h2980996)**.** Let $ABC$ be a triangle with orthocenter $H$ and circumcenter $O$. Let $M$ and $N$ be two points lying on the circumcircle of $\triangle ABC$ such that $MN \parallel BC$. Let $P$ be the point lying on the line $HM$ such that $OP \parallel AN$. Prove that pedal circle of $P$ goes through midpoint of segment $HM$.

**Problem 4** (Diameters)**.** Suppose points $A$, $B$, $P$, and $Q$ lie on a rectangular hyperbola such that $PQ$ passes through the center of the hyperbola, and that $APBQ$ forms a convex quadrilateral in that order. Show that $\angle APB = \angle AQB$.

**Problem 5** (Antigonal Conjugates)$^\star$**.** Let $\triangle ABC$ be a triangle with circumcenter $O$, and let $P$ be a point that is not its orthocenter. Let $Q$ be the reflection of $P$ over the Poncelet point of $P$. Show that if $P^*$ and $Q^*$ are the isogonal conjugates of $P$ and $Q$, respectively, then $OP^* \cdot OQ^* = OA^2$.

**Problem 6** (USA TSTST 2020/6)**.** Let $A$, $B$, $C$, $D$ be four points such that no three are collinear and $D$ is not the orthocenter of $ABC$. Let $P$, $Q$, $R$ be the orthocenters of $\triangle BCD$, $\triangle CAD$, $\triangle ABD$, respectively. Suppose that the lines $AP$, $BQ$, $CR$ are pairwise distinct and are concurrent. Show that the four points $A$, $B$, $C$, $D$ lie on a circle.

The following problems explore some of the named rectangular circumhyperbolae of triangles:

**Problem 7** (Jerabek Hyperbola)**.** The line through the circumcenter $O$ and the orthocenter $H$ is called the **Euler line**; its isogonal conjugate is a rectangular circumhyperbola called the **Jerabek Hyperbola**.

(i) Show that the nine-point circles of $\triangle AOH$, $\triangle BOH$, and $\triangle COH$ all intersect at two common points.

(ii) Let $D$, $E$, and $F$ be the feet of the altitudes from $A$, $B$, and $C$ to the opposite side of $\triangle ABC$. Show that the Euler lines of $\triangle AEF$, $\triangle BDE$, and $\triangle CFD$ concur.

**Problem 8** (Feuerbach Hyperbola)$^{\star}$**.** The **Feuerbach Hyperbola** is the isogonal conjugate of the line through the circumcenter $O$ and the incenter $I$.

(i) Gergonne Point$^{\star}$: Let $D$, $E$, and $F$ be the points at which the incircle is tangent to sides $BC$, $CA$, and $AB$. Show that $AD$, $BE$, and $CF$ concur on the Feuerbach Hyperbola.

(ii) Nagel Point$^{\star}$: Let $D'$, $E'$, and $F'$ be the points at which the $A$-excircle, $B$-excircle, and $C$-excircle are tangent to sides $BC$, $CA$, and $AB$, respectively. Show that $AD'$, $BE'$, and $CF'$ concur on the Feuerbach Hyperbola.

(iii) Kariya's Theorem$^{\star\star}$: Suppose points $X$, $Y$, and $Z$ are in the plane such that $IX \perp BC$, $IY \perp CA$, and $IZ \perp AB$, and that $IX = IY = IZ$. Show that $AX$, $BY$, and $CZ$ concur on the Feuerbach Hyperbola.

**Problem 9** (Kiepert Hyperbola)$^{\star}$**.** The **Kiepert Hyperbola** is the isogonal conjugate of the line through the circumcenter $O$ and the symmedian point $K$ (this is the isogonal conjugate of the centroid $G$.

(i) Kiepert (1869)$^{\star\star}$: Suppose $X$, $Y$, and $Z$ are outside $\triangle ABC$ such that $\triangle BXC$, $\triangle CYA$, and $\triangle AZB$ are similar isosceles triangles. Show that $AX$, $BY$, and $CZ$ concur on the Kiepert Hyperbola.

(ii) Fermat points$^{\star\star\star}$: If we construct equilateral triangles $\triangle BXC$, $\triangle CYA$, and $\triangle AZB$ **outside** $\triangle ABC$, then the concurrency point of the lines $AX$, $BY$, and $CZ$ is called the **First Fermat point** $F^{+}$. Similarly, if we construct equilateral triangles $\triangle BX'C$, $\triangle CY'A$, and $\triangle AZ'B$ pointing **inwards**, then the concurrency point of the lines $AX$, $BY$, and $CZ$ is called the **Second Fermat point** $F^{-}$. Show that the midpoint of segment $F^{+}F^{-}$ is the center of the Kiepert Hyperbola.