THICCC Day 1 Notes

Triangles, Hyperbolas, Isogonal Conjugates, and Certain Circles

Anthony and Nacho

Week 5

The heart of this class lies in the power of circumconics of triangles. Today's class will build up some of the theory required to obtain such power, and discover how things actually relate to circumconics.

Tomorrow, we'll use the power the circumconics give us to get some pretty cool results.

1 Isogonal Conjugates

We start with this definition.

Let's look at pairs of points P and Q such that

$$\angle BAP = \angle CAQ, \quad \angle CBP = \angle ABQ, \quad \text{and} \quad \angle ACP = \angle BCQ.$$

Unfortunately, we run into some issues when one of P or Q is outside the triangle. So instead we define,

> **Definition 1.1.** If P and Q are points, we say that Q is an **isogonal conjugate** of P in triangle ABC if lines AP and AQ are reflections over the angle bisector of $\angle BAC$, lines BP and BQ are reflections over the angle bisector of $\angle CBA$, and lines CP and CQ are reflections over the angle bisector of $\angle ACB$.

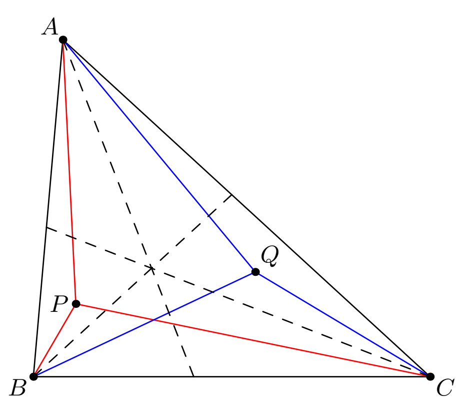

Figure 1: Isogonal conjugates in a triangle

This definition is equivalent the angle condition above. However, this definition works much better when P is outside of the triangle.

One observation we can make is that Q is an isogonal conjugate of P if and only if P is an isogonal conjugate of Q. In fact, this is why we call them "conjugates!"

Proposition 1.2. If P is a point not on the sides of $\triangle ABC$, then P has a unique isogonal conjugate.

To prove this, we appeal to the following theorem:

Theorem 1.3 (Trig Ceva)
Suppose points D, E, and F lie on sides BC, CA, and AB of $\triangle ABC$, respectively. Then AD, BE, and CF concur if and only if

$$\frac{\sin \angle BAD}{\sin \angle CAD} \cdot \frac{\sin \angle CBE}{\sin \angle ABE} \cdot \frac{\sin \angle ACF}{\sin \angle BCF} = 1.$$

Exercise 1. Use Trig Ceva to prove Proposition 1.2.

Proposition 1.4. If P lies on the circumcircle of $\triangle ABC$, then the isogonal conjugate of P is at a "point at infinity."

Proof. More precisely, the reflection of line AP over the angle bisector of $\angle BAC$, the reflection of line BP over the angle biscetor of $\angle CBA$, and the reflection of line CP over the angle bisector of $\angle ACB$ are all parallel (hence, they intersect "infinitely far away").

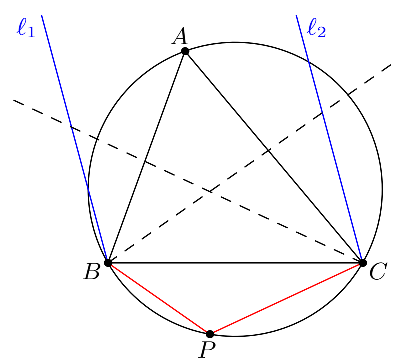

Figure 2: Bottom Text

We will prove that two are parallel; the rest follows by symmetry.

Since $ABCP$ is cyclic, $\angle ABP = 180° - \angle ACP$. However, the angle between the reflection of BP[1] and BC is the same as $\angle ABP$, since the reflection of AB over the angle bisector of $\angle B$ coincides with BC.

Similarly, the angle between the reflection of CP and BC is $\angle ACP$, and it follows that the reflection of CP and BP form supplementary angles with BC, and are thus parallel. As desired. □

[1] over the angle bisector of $\angle B$

2 Pedal Circles

The **pedal** circle of a point P is the circle through the feet of the perpendiculars from P to the sides of $\triangle ABC$.

Theorem 2.1
Isogonal conjugates share the same pedal circle.

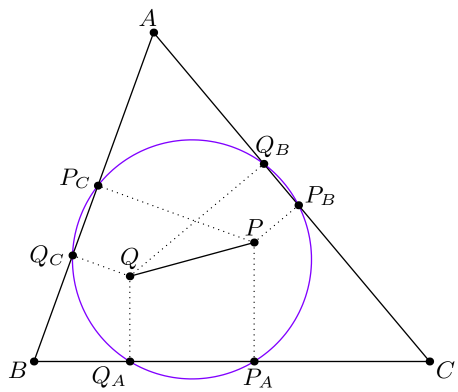

Figure 3: Isogonal Conjugates share a pedal circle.

Proof. Observe that,

$$AP_C \cdot AQ_C = AP \cos\angle BAP \cdot AQ \cos\angle BAQ$$
$$AP_B \cdot AQ_B = AP \cos\angle CAP \cdot AQ \cos\angle CAQ.$$

But by definition, $\angle BAQ = \angle CAP$ and $\angle BAQ = \angle CAQ$. Thus, $AP_C \cdot AQ_C = AP_B \cdot AQ_B$, so P_C, Q_C, P_B, and Q_B are concyclic by the Converse of Power of a Point.

Now, using the same logic, we can prove that P_C, Q_C, P_A, and Q_A are concyclic as well. To prove that all six points are concyclic, construct the three perpendicular bisectors of P_AQ_A, P_BQ_B, and P_CQ_C. All three pass through the midpoint N oy PQ, so this is the center of all six points. □

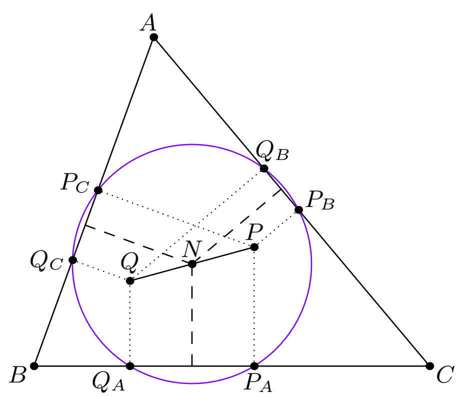

Figure 4: The midpoint of PQ is the center of their shared pedal circle.

Corollary 2.2. *The midpoints of the sides and the feet of the altitudes of a triangle $\triangle ABC$ are concyclic.*

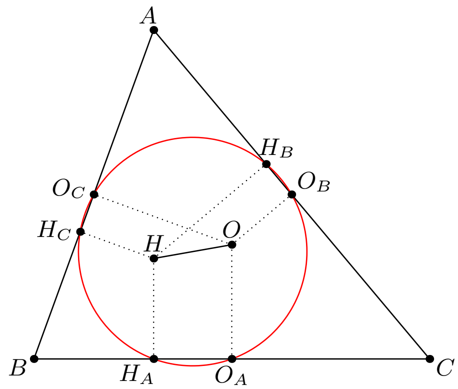

Figure 5: Diagram of Fact 2.2 with the six points labeled.

Why? Because the three midpoints are the feet of the altitudes from the circumcenter O to the sides, and the three feet of the altitudes from the vertices are also the three feet of the altitudes from H to the sides. So this is the pedal triangle of O and H.

This circle is called the **nine-point circle** of $\triangle ABC$.

The **nine-point center** is the center of this circle, and as we noticed earlier, this center should be the midpoint of O and H. Here's another fact about the nine-point circle that comes from some angle chasing:

Fact 2.3. *The reflection of H over the side lengths lies on the circumcircle.*

From this, we can get that the nine-point circle is the result of dilating the circumcircle by a factor of $1/2$ centered at H.

3 Circumconics

Given this isogonal conjugation function whose domain is (almost) the whole plane, one natural question is what the image of a particular curve is under this function. We will consider the isogonal conjugate of a line:

Theorem 3.1
The isogonal conjugate of a line that does not pass through any of the vertices of $\triangle ABC$ is a conic through A, B, and C.

This is called a **circumconic**.

Proof. Place $\triangle ABC$ in $\mathbb{R}^2$. Let $\bullet$ denote arbitrary constants.

Move a point P on the line with constant velocity, so its coordinates are $(\bullet + \bullet t, \bullet + \bullet t)$. If P_B, P_C are the reflections of P across the angle bisectors of B and C, then both P_B and P_C have constant velocity (and hence coordinates of the form $(\bullet + \bullet t, \bullet + \bullet t)$). We wish to show the locus of $BP_B \cap CP_C$ is a conic.

The equation of line BP_B takes the form

$$(\bullet + \bullet t)(x - \bullet) = (\bullet + \bullet t)(y - \bullet),$$

and so does the equation of line CP_C. Solving for t in the above equation gives

$$t = \frac{\bullet x + \bullet y + \bullet}{\bullet_1 x + \bullet_2 y + \bullet_3}$$

where $\bullet_i$ are constants that actually kind of matter. Okay, now substitute this into the equation of line CP_C:

$$\left(\bullet + \frac{\bullet x + \bullet y + \bullet}{\bullet_1 x + \bullet_2 y + \bullet_3}\right)(x - \bullet) = \left(\bullet + \frac{\bullet x + \bullet y + \bullet}{\bullet_1 x + \bullet_2 y + \bullet_3}\right)(y - \bullet)$$

Clear out the denominator; done. □

We also use this general fact from geometry:

Fact 3.2. *Five points in general position determine a conic.*

Thus, two points other than A, B, and C determine a circumconic. Since two points determine a line, every circumconic is the isogonal conjugate of a line that is not one of the sides.

Recalling Proposition 1.4, we see that the circumconic that is the isogonal conjugate of a line ℓ will have points "infinitely far away" if and only if ℓ intersects the circumcircle. In other words,

> **Proposition 3.3.** A line that intersects the circumcircle at two points isogonally conjugates to a circumhyperbola, while a line that does not intersect the circumcircle isogonally conjugates to a circumellipse.

It is also true that a line tangent to the circumcircle isogonally conjugates to a circumparabola, but this is nontrivial.

Since intersections of the line with the circumcircle appears so important to the shape of the conic generated by isogonal conjugation of the line, it is natural to consider what happens when the line passes through the circumcenter (this seems like the most "symmetric" case).

> **Proposition 3.4.** The isogonal conjugate of a line through the circumcenter is a **rectangular circumhyperbola**, which is defined as a circumhyperbola with perpendicular asymptotes.

Proof. Suppose the line hits the circumcircle at X and Y. Then XY is a diameter, so by Thales's theorem, we have $\angle XAY = 90°$. This means that AX and AY are perpendicular, so their reflections over the angle bisector of $\angle BAC$ are perpendicular.

Recalling Proposition 1.4, we know that the isogonal conjugate of points X and Y will correspond with the points at infinity on our conic, which are the asymptotes[2]. □

Corollary 3.5. *The orthocenter H lies on any rectangular circumhyperbola.*

[2] In particular, each asymptote corresponds with the intersection of one of the reflections over an angle bisector and the line at infinity.