THICCC Day 2 Notes

Triangles, Hyperbolas, Isogonal Conjugates, and Certain Circles

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Week 5

1 Circumconics

Recall the following theorem from yesterday:

Theorem 1.1
The isogonal conjugate of a line that does not pass through any of the vertices of $\triangle ABC$ is a conic through A, B, and C.

Using the following fact from geometry, which can be determined via algebra:

Fact 1.2. *Five points in general position determine a conic.*

It follows that two points other than A, B, and C determine a circumconic. Since two points determine a line, every circumconic is the isogonal conjugate of a line that is not one of the sides.

In addition, recall the following theorem:

Theorem 1.3
The isogonal conjugate of a line through the circumcenter is a **rectangular circumhyperbola**, which is a circumhyperbola with perpendicular asymptotes.

Corollary 1.4. *Given four points A, B, C, and D such that D is not the orthocenter of $\triangle ABC$, there is exactly one rectangular hyperbola through all four.*

2 Poncelet Points

Corollary 1.4 is a very symmetric fact. To use this symmetry, we look at the following point:

Definition 2.1. Given a triangle $\triangle ABC$, and a point Q that is not its orthocenter, the **Poncelet point** of Q is the center of the rectangular hyperbola through A, B, C, and Q.

Given this definition[1] , its natural to consider all possible points P on a line through O because the isogonal conjugates Q of all such P lie on a fixed rectangular circumhyperbola, and therefore, have the same Poncelet point.

Now, by lecture theory, we should bring in the nine-point circle and the shared pedal circle of P and Q as well.

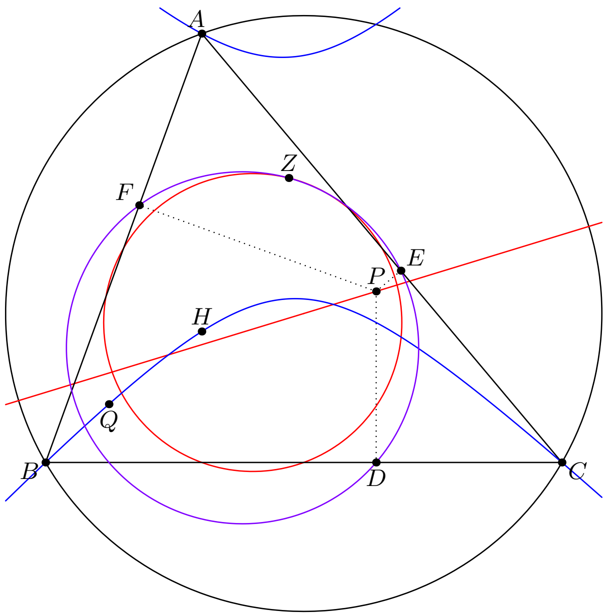

Figure 1: The Poncelet point among our circles (in class, this diagram was interactive).

You might notice the following:

- The Poncelet point of Q appears to lie on the nine-point circle,
- The pedal circle of P and Q appears to pass through the Poncelet point,

We will now prove both.

Theorem 2.2

Given a triangle $\triangle ABC$ and a point Q that is not its orthocenter, the Poncelet Point Z of Q lies on the nine-point circle of $\triangle ABC$.

Proof. We abuse symmetry. Let $\mathcal{H}$ be the rectangular hyperbola through A, B, C, and Q, and let $\mathcal{H}$ hit the circumcircle of $\triangle ABC$ again at D. Note that Z is also the Poncelet point of D.

We know that the orthocenter H of $\triangle ABC$ lies on $\mathcal{H}$. But by symmetry, if H_D is the orthocenter of $\triangle DBC$, then H' lies on $\mathcal{H}$ as well! Finally, note that $AH \parallel DH_D$ because both are perpendicular to BC.

We claim that AHH_DD is actually a parallelogram.

[1] Some authors also define the Poncelet point for any set of four points A, B, C, and Q, which is even more symmetric. We care about circumconics here though, so we will consider them only as a function of one point Q.

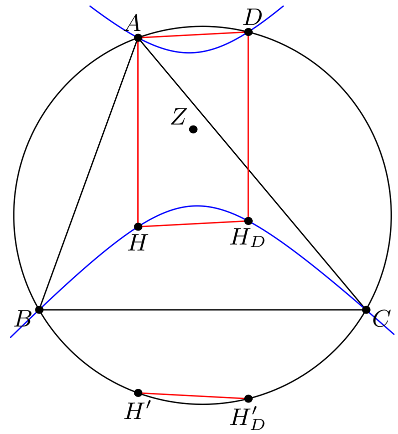

Figure 2: We get a parallelogram!

Why? Well recall from Day 1 that the reflections of H and H_D over line BC lie on the circumcircle of $\triangle ABC$ (which is also the circumcircle of $\triangle DBC$). If we let those reflections be H' and H'_D, respectively, then $HH'H'_DH_D$ is an isosceles trapezoid by reflection, and $AH'H'_DD$ is an isosceles trapezoid since it is cyclic and $AH' \parallel DH'_D$. It follows that

$$\angle HAD = \angle H'AD = \angle AH'H'_D = \angle HH'H'_D = \angle H'HH_D,$$

so $AD \parallel HH_D$.

Thus, parallelogram $AHH'D$ is inscribed in $\mathcal{H}$. But if the center of this parallelogram was not the center of $\mathcal{H}$, then reflecting $\mathcal{H}$ over this center would produce another rectangular hyperbola through A, H, H', and D that is different from $\mathcal{H}$, contradiction.

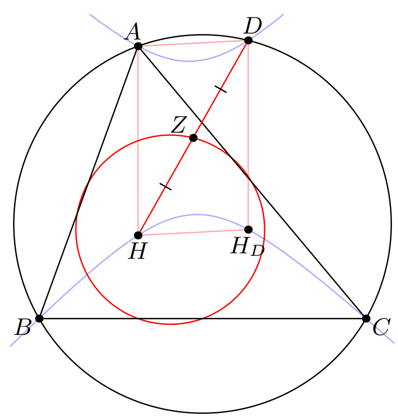

Figure 3: Z is the midpoint of HD

It follows that Z is the center of this parallelogram. But that means that Z is the midpoint of the diagonal HD. But recall that a homothety/scaling by a factor of 1/2 centered at H sends the circumcircle to the nine-point circle. This same scaling sends D to Z, so Z lies on the nine-point circle, as desired. $\square$

Now, we prove the second observation.

Theorem 2.3
Given a triangle $\triangle ABC$ and a point Q that is not its orthocenter, the Poncelet Point Z of Q lies on the pedal circle of Q with respect to $\triangle ABC$.

Proof. We abuse symmetry again; this time, we apply Theorem 2.2 to $\triangle AQB$ and $\triangle AQC$. To access their nine-point circles, let K be the midpoint of AB, let L be the midpoint of AC, and let M be the midpoint of AQ. The theorem implies that Z lies on the circumcircles of both $\triangle KMF$ and $\triangle LME$.

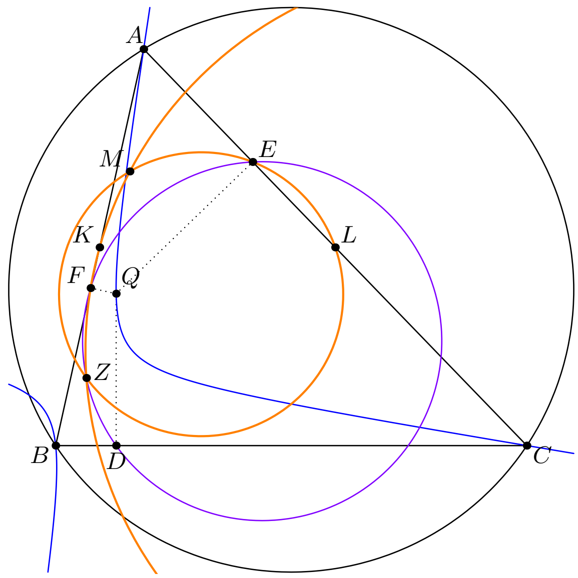

Figure 4: Using nine-point circles to characterize Z

We want to show that $\angle EZF = \angle EDF$. To do this, we split up the former angle into $\angle EZM + \angle FZM$. Now, we have the following chain of angle equalities:

$$\begin{aligned}\angle EZM &= \angle ELM, \quad \text{since } Z \text{ lies on } (LME),\\ &= \angle ALM,\\ &= \angle ACQ, \quad \text{since } \triangle ALM \sim \triangle ACQ \text{ by SAS},\\ &= \angle ECQ,\\ &= \angle EDQ, \quad \text{since } QECD \text{ is cyclic, since } \angle QEC = \angle QDC = 90^\circ.\end{aligned}$$

Similarly, we can show that $\angle FZM = \angle FDQ$. It follows that

$$\angle EZF = \angle EZM + \angle FZM = \angle EDQ + \angle FDQ = \angle EDF,$$

as desired. $\square$

3 Feuerbach's Theorem

Finally, we prove the following theorem from normal circle geometry, as promised in the blurb:

Theorem 3.1 (Feuerbach)
The incircle is tangent to the nine-point circle.

Proof. The key lemma is as follows:

Lemma 3.2
Let P and Q be isogonal conjugates such that line PQ does not pass through the circumcenter O. Furthermore, suppose Z_P and Z_Q are the Poncelet points of P and Q, respectively. Then, Z_P and Z_Q are the two distinct intersections of the nine-point circle and the pedal circle of P and Q.

Proof. Note that lines OP and OQ are different, so their isogonal conjugates are different rectangular hyperbolas. It follows that Z_Q and Z_P are different.

But both lie on the nine-point circle of $\triangle ABC$ and the pedal circle of P and Q, as desired. □

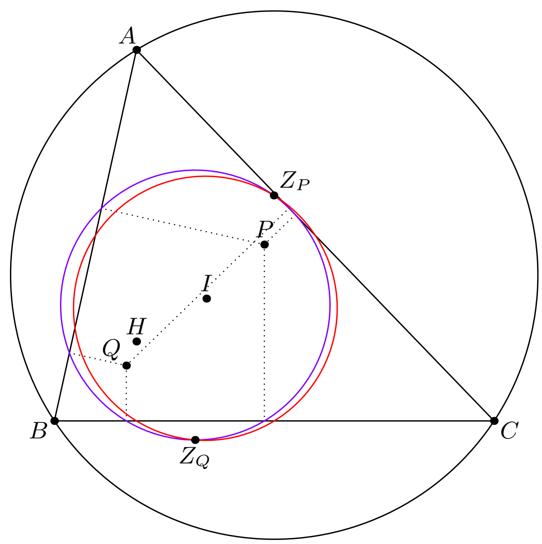

Figure 5: The Poncelet points Z_P and Z_Q are the intersections of the nine-point and pedal circles.

Now, to prove the above theorem, take any point P inside the triangle, and move it toward the incenter I with the following restriction: note that P's isogonal conjugate Q also moves toward I; move P such that P, Q, and O are not always collinear[2].

Since $P \to I$ and $Q \to I$, the two isogonal conjugates P and Q converge to the same point. For continuity reasons, the two Poncelet points Z_P and Z_Q also converge (to the Poncelet point of I).

But this means that on the fixed nine point circle, the two intersections of it with the pedal circle get closer and closer as $P \to I$. At the limit, this means that they are tangent. Finally, the pedal circle of the incenter is the incircle, so we are done. □

[2] In fact, the locus of all points such that P, Q, and O are collinear is a nonsingular cubic curve called the McCay Cubic, so this is a very light restriction.