Trisection Notes

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Here are my notes on my Blair Math Team (half) lecture on angle trisection (on 5/25/2021). First, we discuss the problem of angle trisection, some definitions, and visit Wantzel's proof of the impossibility of angle trisection. Lastly, we look at some methods that we may use to trisect an angle.

1 Trisection Trouble

Angle Trisection is the act of splitting an angle into three congruent angles. The Ancient Greeks were one of the first (known) to propose the problem.

There are currently many methods of trisecting an angle, however, all attempts to trisect an arbitrary angle using only a straightedge and compass were fruitless. The problem remained unsolved.

Many tried to prove that such a construction was impossible, however, the first proof came in the 19th century, which we will look at in a bit.

Problem
Prove that it is impossible to construct a line trisecting any arbitrary angle using only an unmarked straightedge and a compass.

Before we get into the proof, we must first understand the problem itself and the tools required in the proof.

- *Prove*: the goal is to find a completely logical sound argument, which proves the claim.
- *impossible to contsruct*: if it is impossible to construct something, that means we cannot draw something with a certain property with 100% certainty that it is the desired object.
- *arbitrary angle*: This is the important part. This means that there may be some angles which we *can* trisect, but there must be some angle which cannot be trisected.
- *unmarked straightedge*: While we normally use a ruler as a straightedege, we will restrict ourselves to only using the edge of the ruler to draw straight lines, and not the measured numbers.

With that out of the way, let's start by taking the crux of this problem and develop a better understanding of it.

2 Constructions

Let us define the following term:

Definition (Constructable numbers)
Let a number $r \in \mathbb{R}$ be *constructable* if one can draw a line segment with length $|r|$ using a compass and unmarked straightedge a finite number of times, given two marked points in the plane that are 1 unit apart.

It can be shown that this definition is equivalent to a more algebraic statement:

Definition (Constructable numbers, Algebraic)
A number $r \in \mathbb{R}$ is *constructable* if there is a closed form for r using only the numbers 0 and 1, and the operations multiplication, division, subtraction, addition, and square roots.

While this admittedly is nontrivial enough to warrant a proof, the proof will probably be boring and extend past my time.

Remark
If you are wondering how we could get the square root of any constructable number r, consider a circle with diameter $1 + r$ (let's say the endpoints are A and B) and P on this diameter with $AP = 1$. Then the segment perpendicular to AB at P and other endpoint on the circumference has length $\sqrt{r}$, just by Pythagorean theorem or Power of a Point or something.

However, the part we want to remember is the following:

Corollary (Minimal polynomial of a constructible number)
If a number $r \in \mathbb{R}$ is constructible, then the smallest possible degree of a polynomial with rational coefficients and root r must be a power of 2.

The polynomial in question is known as the **minimal polynomial**, but that's a topic for another time. For now, just remember that the least possible degree of a polynomial with root r and rational coefficients must be a power of 2.

3 Proof of Impossibility

This proof is due to Pierre Wantzel, who published this proof in 1837.

First, note that if we can find a single angle that cannot be trisected, we are done, because showing an angle cannot be trisected means that there does not exist a trisection construction in general. Hence, it suffices to prove that a $60°$ angle cannot be trisected with the tools specified.

Note that $\cos 60° = \frac{1}{2}$ is contsructable. Hence, it follows that if a construction exists to trisect $60°$, it implies that $\cos 20°$ must be constructable (since we would be able to construct a $20°$ angle, which means we can construct a right triangle with angle $20°$).

We have the following:

$$\begin{aligned}
\cos 2\theta &= 2\cos^2\theta - 1 \\
\sin 2\theta &= 2\cos\theta\sin\theta \\
\cos(2\theta + \theta) &= \cos 2\theta\cos\theta - \sin 2\theta\sin\theta \\
\cos 3\theta &= 4\cos^3\theta - 3\cos\theta.
\end{aligned}$$

Therefore, $\cos 20°$ must be a root to the polynomial $4x^3 - 3x - \cos 60°$, or $8x^3 - 6x - 1$.

If $8x^3 - 6x - 1$ is reducible over the rationals (in other words, we can split it to be the product of a quadratic and a linear function), it must have a rational root. However, by Rational Root Theorem, the only possible rational roots are $\pm 1, \pm 1/2, \pm 1/4, \pm 1/8$, none of which are actually roots.

Thus, it follows that the minimal polynomial for $\cos 20°$ must be degree 3, implying that $\cos 20°$ is not constructable, and the proof is complete.

Remark

The reason that we use $60°$ is because $\cos 60°$ is rational, and because $20°$ angles are probably not constructable (I say probably because before proving it is not constructable, we wouldn't know this for sure). If we had instead chosen $90°$, this would've failed because $30°$ is a constructable angle, but we couldn't use an angle like $15°$ because $\cos 15°$ is not rational, meaning that we would have to extend our definition of reducibility to include polynomials with irrational coefficients. While that is still doable, we would have to define the notion of a field, and I'm too lazy to do that.

Now that we've proved that we can't trisect an angle with the classical geometry tools, let's see then how we actually can trisect angles!

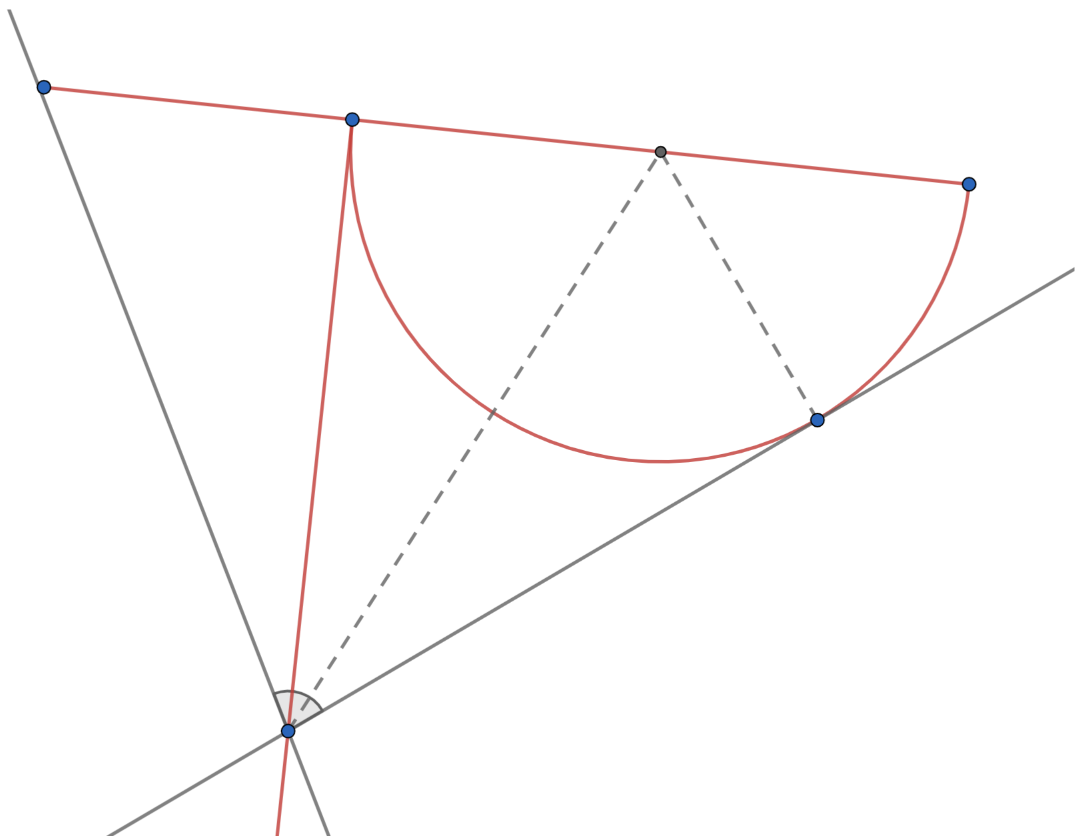
4 Trisection Method 1: Tomahawk

Tools required: paper, scissors, compass, ruler, pencil.

A **tomahawk** is an angle trisection tool I found randomly some few months ago, and I forget exactly how I found it, but in a way it motivated this lecture (that and other things).

I go over how it works in my lecture, but not in these notes because I'm too lazy. Search it up if you are at this point, there are many good videos.

Geogebra demo

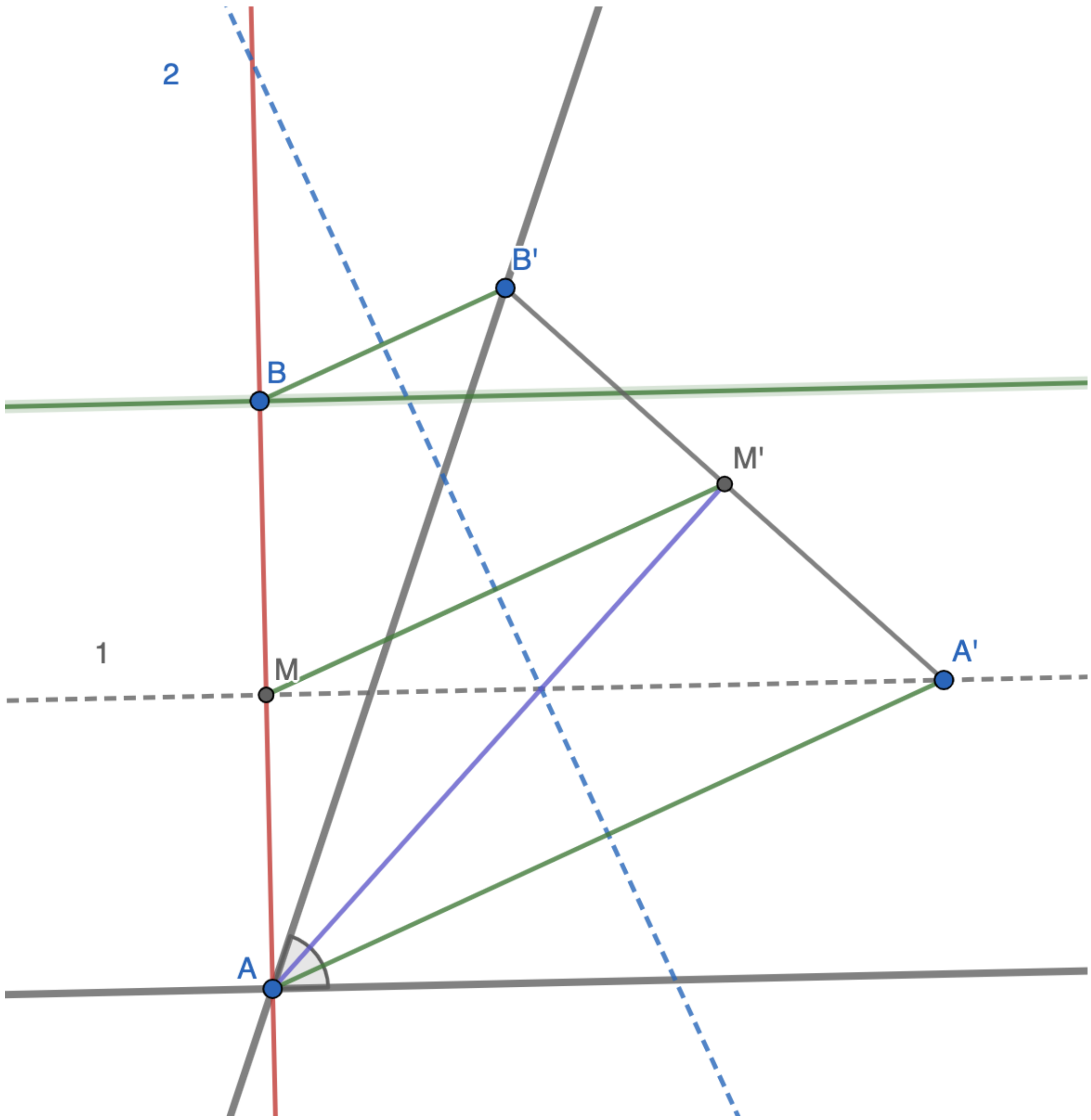


The way it works is quite clever in my opinion. One instantly gets a perpendicular to an iscoceles triangle, which is an angle bisector, alongside of equal tangents giving another angle bisector.

5 Trisection Method 2: Origami

Recently, I have been enlightened by the absolutely unbounded power of origami in math. In fact, it has this extremely simple way to trisect an angle and can be done on any sheet of paper, too. I learned this from this Numberphile video, which only shows the method but no proof.

Geogebra demo



In this diagram, the thick gray lines indicate what we want to trisect, dotted lines are the lines along which we fold.

Since the construction is covered in the video linked above, I'll mainly focus on proof of why this works here.

Proof. Note that B here is an arbitrary point on the perpendicular at A, and M is the midpoint.

When we fold and mark the points, we get A', B', M' which are also reflections of the respective ponits over fold 2. Thus, it follows that $\triangle AM'A \cong \triangle A'MA$.

Since we have MA' is on fold 1, but also parallel to the bottom part, the angle formed by the bottom line and AA' is equal to the angle $\angle MA'A$, and our congruent triangles yields that this angle is equal to $\angle M'AA'$ as well.

Now, note that $BA'A$ is iscoceles with $BA' = AA'$. By congruence, we have that $B'A = AA'$, and thus, $B'AA'$ is iscoceles with altitude AM'.

Thus, we have three equal angles formed from the given angle, as desired. □

Remark

When we fold to place two points onto two lines, this is known as a *Beloch fold*, and has many other uses in (algebraic) geometric origami. For one, when you fold a point onto a line, you are actually creating a crease that is tangent to the parabola with the point as focus and line as directrix, which means that the Beloch fold actually creates a line tangent to two parabolas.